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Stochastic Integral Representation of Bounded Quantum Martingales in Fock Space

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1. INTRODUCTION

It is a classical theorem of Kunita and Watanabe [6] that every square integrable martingale adapted to the standard Brownian motion can be uniquely expressed as the stochastic integral of a nonanticipating square integrable Brownian functional with respect to the same Brownian motion. The central aim of this paper is to establish a similar integral representation of a quantum martingale with respect to the annihilation, creation, and gauge processes in the context of quantum stochastic calculus in Fock space as developed in [1].

In the case of non-Fock quantum Brownian motion such an integral representation was achieved by Hudson and Lindsay [2, 3] with much less difficulty owing to the non-existence of the so called gauge processes.

As special cases of our main result we obtain the differentials of Hilbert-Schmidt and unitary martingales [4]. As an application the uniqueness of Fermion martingales in boson Fock space is established.

The first author wishes to thank Hudson and Lindsay for several useful conversations on this subject.

2. STATEMENT OF THE PROBLEM

In order to make the presentation as self-contained as possible and state our problem precisely we begin with a very brief review of the boson stochastic calculus [1].

For any complex separable Hilbert space $\mathscr{h}$ we denote its inner product by $\langle \cdot, \cdot \rangle$ which is linear on the right and write

$$\Gamma(\mathscr{h}) = \mathbb{C} \oplus \mathscr{h} \oplus \mathscr{h} \circledS \mathscr{h} \oplus \cdots \oplus \mathscr{h}^{\circledS n} \oplus \cdots$$

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where $\mathscr{h}^{\odot n}$ is the n-fold symmetric tensor product of $\mathscr{h}$. We call $\Gamma(\mathscr{h})$ the *boson Fock space* over $\mathscr{h}$. The element

$$\psi(u) = 1 \oplus u \oplus 2^{-1/2} u \otimes u \oplus \cdots \oplus (n!)^{-1/2} u^{\otimes n} \oplus \cdots$$

is called the *exponential* or *coherent vector* associated with $u \in \mathscr{h}$. $\psi(0) = \Omega$ is called the *vacuum vector* in $\Gamma(\mathscr{h})$. For any linear manifold $\mathscr{D} \subset \mathscr{h}$ let $\mathscr{E}(\mathscr{D})$ denote the linear manifold generated by the linearly independent set $\{\psi(u), u \in \mathscr{D}\}$. The *annihilation*, *creation*, and *conservation* (called gauge in [1]) operators are defined on the domain $\mathscr{E}(\mathscr{h})$ respectively by the relations

$$\begin{aligned} a(u)\,\psi(v) &= \langle u, v\rangle\,\psi(v) \\ a^\dagger(u)\,\psi(v) &= \frac{d}{d\varepsilon}\,\psi(v+\varepsilon u)\Big|_{\varepsilon=0} \\ \lambda(T)\,\psi(v) &= \frac{d}{d\varepsilon}\,\psi(e^{\varepsilon T} v)\Big|_{\varepsilon=0} \end{aligned} \tag{2.1}$$

the derivatives being in the strong sense, for all $u, v \in \mathscr{h}$, $T \in \mathscr{B}(\mathscr{h})$, the algebra of all bounded operators on $\mathscr{h}$. The operators $a(u)$ and $a^\dagger(u)$ are adjoint to each other on $\mathscr{E}(\mathscr{h})$. If $T^\dagger$ denotes the adjoint of T then $\lambda(T^\dagger)$ and $\lambda(T)$ are adjoint to each other on $\mathscr{E}(\mathscr{h})$.

We now specialise to the case $\mathscr{h} = L_2(\mathbb{R}_+)$, where $\mathbb{R}_+ = [0, \infty)$. Let $\mathscr{h}_0$ be a fixed complex separable Hilbert space called the *initial space*,

$$\begin{aligned} \mathscr{H} &= \Gamma(L_2(\mathbb{R}_+)), & \tilde{\mathscr{H}} &= \mathscr{h}_0 \otimes \mathscr{H} \\ \tilde{\mathscr{H}}_t &= \mathscr{h}_0 \otimes \Gamma(L_2[0, t]), & \mathscr{H}^t &= \Gamma(L_2[t, \infty)) \end{aligned} \tag{2.2}$$

and let Ω^t, $\Omega^{[a,t]}$ denote the vacuum respectively in the Fock space $\mathscr{H}^t$, $\Gamma(L_2[a, t])$. Observe that

$$\tilde{\mathscr{H}} = \tilde{\mathscr{H}}_t \otimes \mathscr{H}^t \qquad \text{for all} \quad t \geq 0 \tag{2.3}$$

via the unitary correspondence defined by

$$u \otimes \psi(f) \to \{u \otimes \psi(f\,|_{[0,t]})\} \otimes \psi(f\,|_{[t,\infty)}).$$

Let $\mathscr{D}_0 \subset \mathscr{h}_0$ be a dense linear manifold and let $\mathscr{M} \subset L_2(\mathbb{R}_+)$ be a dense linear manifold such that $\chi_{[0,t]} f \in \mathscr{M}$ for all t whenever $f \in \mathscr{M}$. Let

$$\begin{aligned} &\mathscr{M}_t = \{f\,|_{[0,t]}, f \in \mathscr{M}\}, \qquad \mathscr{M}^t = \{f\,|_{[t,\infty)}, f \in \mathscr{M}\} \\ &\tilde{\mathscr{E}} = \mathscr{D}_0 \circledast \mathscr{E}(\mathscr{M}), \qquad \tilde{\mathscr{E}}_t = \mathscr{D}_0 \circledast \mathscr{E}(\mathscr{M}_t), \qquad \mathscr{E}^t = \mathscr{E}(\mathscr{M}^t) \end{aligned} \tag{2.4}$$

where $\circledast$ denotes the algebraic tensor product between vector spaces.

A family of operators $X = \{X(t), t \geqslant 0\}$ is called an *adapted process* with respect to the pair $(\mathscr{D}_0, \mathscr{M})$ if the following conditions are fulfilled:

(i) for all $u \in \mathscr{D}_0$, $f \in \mathscr{M}$,

$$u \otimes \psi(f) \in \mathbb{D}(X(t)), \qquad \text{the domain of } X(t);$$

(ii) $X(t)\, u \otimes \psi(f\chi_{[0,t]})$ is of the form $\tilde{v}_t \otimes \Omega'$ for some $\tilde{v}_t \in \tilde{\mathscr{H}}_t$ and

$$X(t)\, u \otimes \psi(f) = \tilde{v}_t \otimes \psi(f\,|_{[t,\infty)}).$$

An adapted process X is said to be *bounded* if $X(t) = X_t^0 \otimes 1'$ for all $t \geqslant 0$ where $X_t^0 \in \mathscr{B}(\tilde{\mathscr{H}}_t)$ and $1'$ is the identity operator in $\mathscr{H}'$. If, in addition, X_t^0 is Hilbert–Schmidt then X is called a Hilbert–Schmidt adapted process. X is said to be *unitary* if X_t^0 is unitary.

An adapted process X with respect to the pair $(\mathscr{D}_0, \mathscr{M})$ is called a *quantum martingale* or simply a martingale if

$$\begin{aligned}&\langle u \otimes \psi(f\chi_{[0,s]}), X(t)\, v \otimes \psi(g\chi_{[0,s]})\rangle \\ &\quad = \langle u \otimes \psi(f\chi_{[0,s]}), X(s)\, v \otimes \psi(g\chi_{[0,s]})\rangle \end{aligned} \tag{2.5}$$

for all $0 \leqslant s < t < \infty$, $u, v \in \mathscr{D}_0$, $f, g \in \mathscr{M}$.

Since $\mathscr{D}_0$ is dense in h_0 and $\mathscr{M}$ is dense in $L_2(\mathbb{R}_+)$ it follows that $\tilde{\mathscr{E}}_t$ is dense in $\tilde{\mathscr{H}}_t$ for each t and hence the boundedness of a martingale X implies the identity

$$\langle \tilde{u}_s, X(t)\, \tilde{v}_s\rangle = \langle \tilde{u}_s, X(s)\, \tilde{v}_s\rangle \qquad \text{if} \quad s < t,\ \tilde{u}_s, \tilde{v}_s \in \tilde{\mathscr{H}}_s. \tag{2.6}$$

Following [1] we introduce three basic martingales called *annihilation*, *creation*, and *conservation* respectively with respect to the pair $(h_0, L_2(\mathbb{R}_+))$:

$$\begin{aligned} A(t) &= 1_0 \otimes a(\chi_{[0,t]}) \\ A^\dagger(t) &= 1_0 \otimes a^\dagger(\chi_{[0,t]}) \\ \Lambda(t) &= 1_0 \otimes \lambda(\chi_{[0,t]}) \end{aligned}$$

on the domain $\tilde{\mathscr{E}}$ where a, $a^\dagger$, λ are defined by (2.1), $\chi_{[0,t]}$ is the element in $L_2(\mathbb{R}_+)$ in the first two equations and the projection operator of multiplication by $\chi_{[0,t]}$ in $L_2(\mathbb{R}_+)$ in the last equation.

Let $\mathscr{A}_2(\mathscr{D}_0, \mathscr{M})$ denote the class of all ordered quadruples of adapted processes $\{E_j(t), 1 \leqslant j \leqslant 4, t \geqslant 0\} = \mathbf{E}$ satisfying the inequalities:

$$\int_0^t \left\{ |f(s)|^2\, \|E_1(s)\, u \otimes \psi(f)\|^2 + \sum_{j=2}^4 \|E_j(s)\, u \otimes \psi(f)\|^2 \right\} ds < \infty$$

for all $t>0$, $u \in \mathscr{D}_0$, $f \in \mathscr{M}$. For any $\mathbf{E} \in \mathscr{A}_2(\mathscr{D}_0, \mathscr{M})$ we consider the *stochastic integral* [1] defined by

$$X(t)=\int_0^t E_1\, d\Lambda + E_2\, dA + E_3\, dA^\dagger + E_4\, ds$$

where X is an adapted process with respect to $(\mathscr{D}_0, \mathscr{M})$ satisfying the relation

$$\langle u \otimes \psi(f), X(t)\, v \otimes \psi(g)\rangle = \int_0^t \langle u \otimes \psi(f), (\bar{f}(s)\, g(s)\, E_1(s) + g(s)\, E_2(s) + \bar{f}(s)\, E_3(s) + E_4(s))\, v \otimes \psi(g)\rangle\, ds \tag{2.7}$$

for all u, $v \in \mathscr{D}_0$, f, $g \in \mathscr{M}$.

If X is an adapted process satisfying the relation

$$X(t) = X(0) + \int_0^t E_1\, d\Lambda + E_2\, dA + E_3\, dA^\dagger + E_4\, ds$$

for some $\mathbf{E} \in \mathscr{A}_2(\mathscr{D}_0, \mathscr{M})$ then we write

$$dX = E_1\, d\Lambda + E_2\, dA + E_3\, dA^\dagger + E_4\, dt. \tag{2.8}$$

If Y is another adapted process such that

$$dY = F_1\, d\Lambda + F_2\, dA + F_3\, dA^\dagger + F_4\, dt$$

where $\mathbf{F} \in \mathscr{A}_2(\mathscr{D}'_0, \mathscr{M}')$ then we have the *quantum Ito product formula* [1]: for all $u \in \mathscr{D}_0$, $v \in \mathscr{D}'_0$, $f \in \mathscr{M}$, $g \in \mathscr{M}'$,

$$\begin{aligned}
&\langle X(t)\, u \otimes \psi(f), Y(t)\, v \otimes \psi(g)\rangle \\
&\quad = \langle X(0)\, u \otimes \psi(f), Y(0)\, v \otimes \psi(g)\rangle \\
&\qquad + \int_0^t \{\langle X(s)\, u \otimes \psi(f), (\bar{f}(s)\, g(s)\, F_1(s) \\
&\qquad + g(s)\, F_2(s) + \bar{f}(s)\, F_3(s) + F_4(s))\, v \otimes \psi(g)\rangle \\
&\qquad + \langle (f(s)\, \overline{g(s)}\, E_1(s) + f(s)\, E_2(s) \\
&\qquad + \bar{g}(s)\, E_3(s) + E_4(s)) u \otimes \psi(f), Y(s) v \otimes \psi(g)\rangle \\
&\qquad + \bar{f}(s)\, g(s) \langle E_1(s)\, u \otimes \psi(f), F_1(s)\, v \otimes \psi(g)\rangle \\
&\qquad + \bar{f}(s) \langle E_1(s)\, u \otimes \psi(f), F_3(s)\, v \otimes \psi(g)\rangle \\
&\qquad + g(s) \langle E_3(s)\, u \otimes \psi(f), F_1(s)\, v \otimes \psi(g)\rangle \\
&\qquad + \langle E_3(s)\, u \otimes \psi(f), F_3(s)\, v \otimes \psi(g)\rangle\}\, ds.
\end{aligned} \tag{2.9}$$

The identities (2.7) and (2.9) constitute the foundations of quantum stochastic calculus and all our present analysis.

If $E_4 = 0$ in (2.8) it follows from (2.7) that X is a martingale. Our goal is to study the converse of this property. That the converse is not necessarily true is shown by the following counterexample due to Journé and Meyer [6]: let $\mathscr{h}_0 = \mathbb{C} = \mathscr{D}_0$, $\mathscr{M} = L_2(\mathbb{R}_+)$ and $X(t)$ is the operator defined on $\tilde{\mathscr{E}}$ by the relation $X(t)\psi(f) = \psi(C_t f)$, $C_t f = \chi_{[0,t]} H \chi_{[0,t]} f$, H denoting the Hilbert transform restricted to $L_2(\mathbb{R}_+)$. Then $X(t)$ is a bounded martingale with $X(0) = 0$ but X cannot be expressed in the form $dX = E_1\, d\Lambda + E_2\, dA + E_3\, dA^+$. In the next section we shall consider the case of bounded martingales obeying a natural regularity condition and show that they admit a representation (2.8) with $E_4 = 0$.

3. Regular Martingales

In the sequel we shall identify the Hilbert space $\tilde{\mathscr{H}}_t$ occurring in the factorisation of $\tilde{\mathscr{H}}$ given by (2.2) and (2.3) with the subspace $\tilde{\mathscr{H}}_t \otimes \Omega^t \subset \tilde{\mathscr{H}}$ through the isomorphism $\tilde{u} \to \tilde{u} \otimes \Omega^t$, $\tilde{u} \in \tilde{\mathscr{H}}_t$. Similarly, for $t > a > 0$ we shall identify the Hilbert spaces $\tilde{\mathscr{H}}_a$ and $\tilde{\mathscr{H}}_a \otimes \Omega^{[a,t]}$.

We shall make frequent use of the existence of the natural isometric isomorphism between $\tilde{\mathscr{H}}$ and $L_2(\mathscr{h}_0, \omega)$, the Hilbert space of $\mathscr{h}_0$-valued square integrable random variables with respect to the Wiener measure ω of standard Brownian motion w, via the correspondence

$$u \otimes \psi(f) \to u \exp\left(\int_0^\infty f(s)\, dw(s) - \frac{1}{2} \int_0^\infty f(s)^2\, ds\right),$$

$u \in \mathscr{h}_0$, $f \in L_2(\mathbb{R}_+)$. With this correspondence in view we write the following stochastic equation:

$$u \otimes \psi(f\chi_{[0,t]}) = u \otimes \Omega + \int_0^t f(s)\, u \otimes \psi(f\chi_{[0,s]})\, dw(s). \tag{3.1}$$

The symbol $\mathbb{E}$ will denote expectation with respect to the Wiener measure ω.

We need the following simple adaptation of the Kunita Watanabe theorem for the case of vector valued martingales:

Proposition 3.1. *Let $\mathscr{k}$ be any complex separable Hilbert space and let $\{x(t)\}$ be a $\mathscr{k}$-valued square integrable classical martingale adapted to the standard Brownian motion w, so that $\mathbb{E}\, |X(t)|^2 < \infty$ for all t. Then there* *exists a unique k-valued nonanticipating square integrable Brownian functional ξ such that*

$$X(t) = X(0) + \int_0^t \xi(s, w)\, dw(s).$$

Proof. This is immediate from the scalar version of the Kunita Watanabe theorem applied to each of the coordinate martingales when k is identified with l_2. ∎

It may be noted that every k-valued square integrable martingale adapted to the Brownian motion w is automatically mean square continuous in the time variable.

DEFINITION. A bounded martingale X on $\tilde{\mathscr{H}}$ is said to be *regular with respect to a Radon measure* μ on $[0, \infty)$, or simply *regular* if for all $t > a \geqslant 0$, $\tilde{u} \in \tilde{\mathscr{H}}_a$,

$$\max(\|[X(t) - X(a)]\, \tilde{u}\|^2, \|[X^\dagger(t) - X^\dagger(a)]\, \tilde{u}\|^2) \leqslant \|\tilde{u}\|^2\, \mu([a, t]). \quad (3.2)$$

The next proposition shows that the regularity property is a necessary condition for a large class of bounded martingales admitting the representation (2.8) with $E_4 = 0$.

PROPOSITION 3.2. *Let X be a bounded martingale on $\tilde{\mathscr{H}}$ with the representation*

$$dX = M\, dA + K^\dagger\, dA + L\, dA^\dagger$$

where the quadruples $(M, K^\dagger, L, 0)$, $(M^\dagger, L^\dagger, K, 0) \in \mathscr{A}_2(k_0, \mathscr{M})$, M, K, L are bounded adapted processes and $\|K(t)\|$ and $\|L(t)\|$ are locally square integrable in t. Then X is regular.

Proof. From the definition of a bounded martingale and (2.7) it follows that $\{X^\dagger(t)\}$ is a bounded martingale with the representation

$$dX^\dagger = M^\dagger\, dA + L^\dagger\, dA + K\, dA^\dagger.$$

By (2.6), for any $t > a \geqslant 0$, $\tilde{u} \in \tilde{\mathscr{H}}_a$, we have

$$\begin{aligned} \|[X(t) - X(a)]\, \tilde{u}\|^2 &= \|X(t)\, \tilde{u}\|^2 - \|X(a)\, \tilde{u}\|^2 \\ \|[X^\dagger(t) - X^\dagger(a)]\, \tilde{u}\|^2 &= \|X^\dagger(t)\, \tilde{u}\|^2 - \|X^\dagger(a)\, \tilde{u}\|^2. \end{aligned} \quad (3.3)$$

An application of Ito's product formula (2.9) shows that for all $u, v \in h_0$, $\tilde{u} \in \tilde{\mathscr{E}}_a$ (defined by (2.4) when $\mathscr{D}_0 = h_0$, $\mathscr{M} = L_2(\mathbb{R}_+)$), $t > a \geqslant 0$,

$$\|X(t)\,\tilde{u}\|^2 - \|X(a)\,\tilde{u}\|^2 = \int_a^t \|L(s)\,\tilde{u}\|^2\,ds \leqslant \|\tilde{u}\| \int_a^t \|L(s)\|^2\,ds.$$

Similarly,

$$\|X^\dagger(t)\,\tilde{u}\|^2 - \|X^\dagger(a)\,\tilde{u}\|^2 \leqslant \|\tilde{u}\|^2 \int_a^t \|K(s)\|^2\,ds.$$

Setting

$$\mu([a, b]) = \int_a^b (\|L(s)\|^2 + \|K(s)\|^2)\,ds \qquad \text{for all } 0 \leqslant a \leqslant b < \infty,$$

using (3.3) and the density of $\tilde{\mathscr{E}}_a$ in $\tilde{\mathscr{H}}_a$, we obtain the regularity of X with respect to the absolutely continuous Radon measure μ. ∎

In order to achieve our main goal of establishing the converse of Proposition 3.2 we prove a series of propositions.

PROPOSITION 3.3. *Let X be a bounded martingale regular with respect to a Radon measure μ on $\mathbb{R}_+$. Then* (i) $\|X(t)\|$ *is nondecreasing and* (ii) *the inequality* (3.2) *is satisfied with μ replaced by its absolutely continuous part* μ_{ac}.

Proof. (i) Since (2.6) implies (3.3) and $X(t) = X_t^0 \otimes 1'$ we have for any $t > a$,

$$\|X(t)\| \geqslant \sup_{\tilde{u} \in \tilde{\mathscr{H}}_a,\, \|\tilde{u}\| = 1} \|X(t)\,\tilde{u}\| \geqslant \|X_a^0\| = \|X(a)\|.$$

(ii) Observe that for any fixed $\tilde{u} \in \tilde{\mathscr{H}}_a$, $\{X(t)\,\tilde{u}, t \geqslant a\}$ and $\{X^\dagger(t)\,\tilde{u}, t \geqslant a\}$ are classical h_0-valued square integrable martingales in $[a, \infty)$ adapted to Brownian motion and hence by Proposition 3.1 there exist h_0-valued nonanticipating Brownian functionals $\xi(t, \tilde{u})$, $\eta(t, \tilde{u})$, $t \geqslant a$ such that

$$X(t)\,\tilde{u} - X(a)\,\tilde{u} = \int_a^t \xi(s, \tilde{u})\,dw(s),$$

$$X^\dagger(t)\,\tilde{u} - X^\dagger(a)\,\tilde{u} = \int_a^t \eta(s, \tilde{u})\,dw(s).$$

By the isometry of Ito integrals and (3.2) we have for all $0 \leqslant a \leqslant b < t < \infty$

$$\begin{aligned}\max\left(\int_b^t \mathbb{E}\,|\xi(s, \tilde{u})|^2\,ds, \int_b^t \mathbb{E}\,|\eta(s, \tilde{u})|^2\,ds\right) &= \max(\|[X(t) - X(b)]\,\tilde{u}\|^2, \|[X^\dagger(t) - X^\dagger(b)]\,\tilde{u}\|^2) \\ &\leqslant \|\tilde{u}\|^2\,\mu[b, t])\end{aligned}$$

where $|\cdot|$ denotes the norm in $\mathscr{k}_0$. This shows that μ in (3.2) can be replaced by μ_{ac}. ∎

By virtue of (ii) in Proposition 3.3 we can assume without loss of generality that μ is an absolutely continuous Radon measure.

PROPOSITION 3.4. *Let X be a bounded martingale regular with respect to an absolutely continuous Radon measure μ on $\mathbb{R}_+$ with density $\mu'(t)$. Then there exists a bounded adapted process $\{L(t)\}$ such that*

(i) $[X(t) - X(a)]\,\tilde{u} = \int_a^t L(s)\,\tilde{u}\,dw(s)$ *for all* $t > a$, $\tilde{u} \in \tilde{\mathscr{H}}_a$

(ii) $\|L(s)\|^2 \leqslant \mu'(s)$ *for all* s.

Proof. Fix $a > 0$, $\tilde{u} \in \tilde{\mathscr{H}}_a$. As in the proof of Proposition 3.3(ii) we conclude the existence of a $\mathscr{k}_0$-valued square integrable nonanticipating Brownian functional $\xi_a(\cdot, \tilde{u})$ in $[a, \infty)$ such that

$$[X(t) - X(a)]\,\tilde{u} = \int_a^t \xi_a(s, \tilde{u})\,dw(s), \qquad t > a. \tag{3.4}$$

For $b > a$, $\tilde{u}$ can be thought of as belonging to $\tilde{\mathscr{H}}_b$ and

$$[X(t) - X(b)]\,\tilde{u} = \int_b^t \xi_b(s, \tilde{u})\,dw(s), \qquad t > b. \tag{3.5}$$

On the other hand, by (3.4) we have

$$\begin{aligned}[X(t) - X(b)]\,\tilde{u} &= [X(t) - X(a)]\,\tilde{u} - [X(b) - X(a)]\,\tilde{u} \\ &= \int_b^t \xi_a(s, \tilde{u})\,dw(s).\end{aligned}$$

By the uniqueness of the representation in Proposition 3.1, (3.4) and (3.5) imply $\xi_a(s, \tilde{u}) = \xi_b(s, \tilde{u})$ a.e. $s > b > a$. Hence we can define

$$\xi(s, \tilde{u}) = \xi_a(s, \tilde{u}) \qquad \text{for} \quad s > a,\ \tilde{u} \in \tilde{\mathscr{H}}_a.$$

By (3.2) and isometry of Ito integral we have

$$\int_b^t \mathbb{E}\,|\xi(s,\tilde{u})|^2\,ds = \|[X(t)-X(\mathrm{b})]\,\tilde{u}\|^2 \leqslant \|\tilde{u}\|^2\,\mu([b,t]). \tag{3.6}$$

On identifying the $\mathscr{h}_0$-valued classical stochastic process $\xi(s,\tilde{u})$ with the vector $\xi(s,\tilde{u})\in\tilde{\mathscr{H}}_s$, $s>a$ we conclude from (3.6) that

$$\|\xi(s,\tilde{u})\|^2 \leqslant \|\tilde{u}\|^2\,\mu'(s) \qquad \text{a.e.} \quad s>a. \tag{3.7}$$

Now choose and fix countable dense sets $D_0\subset \mathscr{h}_0$, $D\subset L_2(\mathbb{R}_+)$ and let γ denote a typical index $(r_1, r_2,..., r_k;\ u_1,..., u_k;\ f^{(1)}, f^{(2)},..., f^{(k)})$ where r_j's are complex numbers with rational real and imaginary parts, $u_j\in D_0$, $f^{(j)}\in D$, $k=1, 2,...$ so that γ varies in a countable set $\mathscr{I}$. Define

$$\begin{aligned}\phi(a,s,\gamma) &= 0 && \text{if} \quad s\leqslant a\\ &= \left\|\sum_1^k r_j\xi(s, u_j\otimes\psi(f^{(j)}\chi_{[0,a]}))\right\|^2 \\ &\quad -\mu'(s)\left\|\sum_1^k r_ju_j\otimes\psi(f^{(j)}\chi_{[0,a]})\right\|^2 && \text{if} \quad s>a.\end{aligned}$$

Since $\tilde{u}\to\xi(s,\tilde{u})$ is linear it follows from (3.7) and the definition of ϕ that $\{s: \phi(a,s,\gamma)>0\}$ has Lebesgue measure 0 for every positive a and every $\gamma\in\mathscr{I}$. In other words there exists a null set $F\subset[0,\infty)$ such that

$$\left\|\sum_{j=1}^k r_j\xi(s, u_j\otimes\psi(f^{(j)}\chi_{[0,a]}))\right\|^2 \leqslant \mu'(s)\left\|\sum_{j=1}^k r_ju_j\otimes\psi(f^{(j)}\chi_{[0,a]})\right\|^2 \tag{3.8}$$

for all positive rational a, $s\notin F$, $s>a$, $\gamma\in\mathscr{I}$. Now set

$$\begin{aligned}L(s)\,u\otimes\psi(f\chi_{[0,a]}) &= \xi(s, u\otimes\psi(f\chi_{[0,a]})) && \text{if} \quad s\notin F,\\ &= 0 && \text{otherwise}\end{aligned}$$

for rational $a>0$, $s>a$, $u\in D_0$, $f\in D$ and extend linearly. Then (3.8) along with the density of the sets $\{\sum_{j=1}^k r_j\xi(s, u_j\otimes\psi(f^{(j)}\chi_{[0,a]}),\ \gamma\in\mathscr{I}\}$ and $\bigcup_{0<a<s,\,a\,\text{rational}}\tilde{\mathscr{H}}_a$ in $\tilde{\mathscr{H}}_a$ and $\tilde{\mathscr{H}}_s$ respectively implies the existence of a bounded operator $L(s)$ on $\tilde{\mathscr{H}}_s$ such that for each $\tilde{u}\in\tilde{\mathscr{H}}_a$,

$$L(s)\,\tilde{u} = \xi(s,\tilde{u}) \qquad \text{a.e. } s>a$$

and

$$\|L(s)\|^2 \leqslant \mu'(s) \qquad \text{for all } s.$$

Finally, in the factorisation $\tilde{\mathscr{H}} = \tilde{\mathscr{H}}_s \otimes \mathscr{H}^s$ set $L(s)$ as $L(s) \otimes 1^s$. Then we obtain (i) from (3.4). ∎

COROLLARY 3.5. *Let X be as in Proposition 3.4. Then there exists a bounded adapted process $\{K(t)\}$ such that for all $t > a$, $\tilde{u} \in \tilde{\mathscr{H}}_a$*

$$[X^\dagger(t) - X^\dagger(a)]\, \tilde{u} = \int_a^t K(s)\, \tilde{u}\, dw(s)$$

where

$$\|K(t)\|^2 \leqslant \mu'(t) \qquad \text{for all } t.$$

Proof. Since $X^\dagger$ is also a bounded regular martingale this is immediate from Proposition 3.4. ∎

Remark. It is clear from the proofs of Proposition 3.4 and Corollary 3.5 that the operator adapted processes L and K are *uniquely* determined (modulo a Lebesgue null set) by X and $X^\dagger$, respectively. In the following we shall write $K = K_X$, $L = L_X$ whenever necessary in order to avoid confusion.

PROPOSITION 3.6. *Let X be as in Proposition 3.4 and let $\tilde{\mathscr{E}}$ be defined by (2.4) with $\mathscr{D}_0 = h_0$, $\mathscr{M} = L_2(\mathbb{R}_+)$. Define the operators $S(t)$, $S^\dagger(t)$, $Z(t)$, $Z^\dagger(t)$ on $\tilde{\mathscr{E}}$ by*

$$S(t) = \int_0^t K_X^\dagger(s)\, dA(s) + \int_0^t L_X(s)\, dA^\dagger(s),$$

$$S^\dagger(t) = \int_0^t L_X^\dagger(s)\, dA(s) + \int_0^t K_X(s)\, dA^\dagger(s),$$

$$Z(t) = X(t) - S(t),\ Z^\dagger(t) = X^\dagger(t) - S^\dagger(t).$$

Then (i) $\{S, S^\dagger\}$, $\{Z, Z^\dagger\}$ are adjoint pairs on $\tilde{\mathscr{E}}$; and for any $u \in h_0$, $f \in L_2(\mathbb{R}_+)$,

(ii) $\{Z(t)\, u \otimes \psi(f\chi_{[0,t]})\}$ and $\{Z^\dagger(t)\, u \otimes \psi(f\chi_{[0,t]})\}$ are classical h_0-valued martingales adapted to Brownian motion;

(iii) there exist h_0-valued square integrable classical processes $\xi(\cdot, u, f)$ and $\eta(\cdot, u, f)$ adapted to Brownian motion such that

$$Z(t)\, u \otimes \psi(f\chi_{[0,t]}) = X(0)\, u \otimes \Omega + \int_0^t f(s)\, \xi(s, u, f)\, dw(s),$$

$$Z^\dagger(t)\, u \otimes \psi(f\chi_{[0,t]}) = X^\dagger(0)\, u \otimes \Omega + \int_0^t f(s)\, \eta(s, u, f)\, dw(s);$$

(iv) $Z(t)\,u\otimes\psi(f\chi_{[0,a]})=Z(a)\,u\otimes\psi(f\chi_{[0,a]})$,

$$Z^{\dagger}(t)\,u\otimes\psi(f\chi_{[0,a]})=Z^{\dagger}(a)\,u\otimes\psi(f\chi_{[0,a]})\qquad\text{for}\quad t\geqslant a.$$

Proof. (i) By (2.7) and the estimates of Proposition 3.4 and Corollary 3.5 it follows that $\{S, S^{\dagger}\}$ and $\{Z, Z^{\dagger}\}$ are well-defined adapted processes on $\tilde{\mathscr{E}}$ which are easily seen to be adjoint pairs on $\tilde{\mathscr{E}}$.

(ii) By Corollary 3.5, (3.1), and the isometry of Ito integrals it follows that for $t>a$, $\tilde{v}\in\tilde{\mathscr{H}}_a$,

$$\begin{aligned}
&\langle[X(t)-X(a)]\,u\otimes\psi(f\chi_{[0,t]}),\tilde{v}\rangle\\
&\quad=\langle u\otimes\psi(f\chi_{[0,t]}),[X^{\dagger}(t)-X^{\dagger}(a)]\,\tilde{v}\rangle\\
&\quad=\int_a^t\bar{f}(s)\langle u\otimes\psi(f\chi_{[0,s]}),K_X(s)\,\tilde{v}\rangle\,ds\\
&\quad=\left\langle\int_a^t K_X^{\dagger}(s)\,dA(s)\,u\otimes\psi(f\chi_{[0,t]}),\tilde{v}\right\rangle.
\end{aligned}\tag{3.9}$$

For $v\in\mathscr{k}_0$, $g\in L_2(\mathbb{R}_+)$ we have

$$\left\langle\left\{\int_a^t L_X(s)\,dA^{\dagger}(s)\right\}u\otimes\psi(f\chi_{[0,t]}),v\otimes\psi(g\chi_{[0,a]})\right\rangle=0$$

for $t>a$. This together with (3.9) implies that $\{Z(t)\,u\otimes\psi(f\chi_{[0,t]})\}$ is a classical martingale. Similarly $\{Z^{\dagger}(t)\,u\otimes\psi(f\chi_{[0,t]})\}$ is a classical martingale.

(iii) By (ii) and Proposition 3.1 there exist nonanticipating Brownian functionals $\xi_1(\cdot,u,f)$, $\eta_1(\cdot,u,f)$ such that

$$Z(t)\,u\otimes\psi(f\chi_{[0,t]})=X(0)\,u\otimes\Omega+\int_0^t\xi_1(s,u,f)\,dw(s),$$

$$Z^{\dagger}(t)\,u\otimes\psi(f\chi_{[0,t]})=X^{\dagger}(0)\,u\otimes\Omega+\int_0^t\eta_1(s,u,f)\,dw(s).$$

By (3.1) and isometry of Ito integrals we have for $v\in\mathscr{k}_0$, $g\in L_2(\mathbb{R}_+)$

$$\begin{aligned}
&\langle v\otimes\psi(g\chi_{[0,t]}),Z(t)\,u\otimes\psi(f\chi_{[0,t]})\rangle\\
&\quad=\langle v\otimes\Omega,X(0)\,u\otimes\Omega\rangle+\int_0^t\bar{g}(s)\langle v\otimes\psi(g\chi_{[0,s]}),\xi_1(s,u,f)\rangle\,ds\\
&\quad=\langle Z^{\dagger}(t)\,v\otimes\psi(g\chi_{[0,t]}),u\otimes\psi(f\chi_{[0,t]}\rangle\\
&\quad=\langle X^{\dagger}(0)\,v\otimes\Omega,u\otimes\Omega\rangle+\int_0^t f(s)\langle\eta_1(s,v,g),u\otimes\psi(f\chi_{[0,s]})\rangle\,ds.
\end{aligned}$$

Thus

$$\bar{g}(s)\langle v\otimes\psi(g\chi_{[0,s]}), \xi_1(s,u,f)\rangle = f(s)\langle \eta_1(s,v,g), u\otimes\psi(f\chi_{[0,s]})\rangle$$

a.e. s. Fix f and let $N=\{s: f(s)=0\}$. Then for every nowhere vanishing function $g\in L_2(\mathbb{R}_+)$ we have

$$\langle v\otimes\psi(g\chi_{[0,s]}), \xi_1(s,u,f)\rangle = 0 \qquad \text{a.e.} \quad s\in N.$$

Varying v over a countable total set in $\mathscr{h}_0$ and g over a countable dense set of nowhere vanishing functions in $L_2(\mathbb{R}_+)$ and using the totality of all such vectors $v\otimes\psi(g\chi_{[0,s]})$ in $\tilde{\mathscr{H}}_s$ we conclude

$$\xi_1(s,u,f)=0 \qquad \text{a.e.} \quad s\in N.$$

Let

$$\begin{aligned} \xi(s,u,f) &= f(s)^{-1}\,\xi_1(s,u,f) &&\text{if} \quad f(s)\neq 0,\\ &= 0 &&\text{if} \quad f(s)=0. \end{aligned}$$

By the isometry of Ito integrals and Fubini's theorem we conclude that $\mathbb{E}\,|\xi(s,u,f)|^2<\infty$. Similarly $\eta_1(s,v,g)=g(s)\,\eta(s,v,g)$, where $\mathbb{E}\,|\eta(s,v,g)|^2<\infty$.

(iv) Let $v\in\mathscr{h}_0$, $g\in L_2(\mathbb{R}_+)$, $t>a$. By (i) and (ii),

$$\begin{aligned} &\langle v\otimes\psi(g\chi_{[0,t]}), Z(t)\,u\otimes\psi(f\chi_{[0,a]})\rangle\\ &\quad = \langle Z^\dagger(t)\,v\otimes\psi(g\chi_{[0,t]}), u\otimes\psi(f\chi_{[0,a]})\rangle\\ &\quad = \langle Z^\dagger(a)\,v\otimes\psi(g\chi_{[0,a]}), u\otimes\psi(f\chi_{[0,a]})\rangle\\ &\quad = \langle v\otimes\psi(g\chi_{[0,t]}), Z(a)\,u\otimes\psi(f\chi_{[0,a]})\rangle. \end{aligned}$$

Since $\{v\otimes\psi(g\chi_{[0,t]}),\ v\in\mathscr{h}_0,\ g\in L_2(\mathbb{R}_+)\}$ is total in $\tilde{\mathscr{H}}_t$ we have the first part of (iv). Its second part follows similarly. ∎

We shall now show that Z is a stochastic integral with respect to the martingale A. To this end we use a special martingale U which is related to the Weyl representation (See Sects. 2 and 6 in [1]).

PROPOSITION 3.7. *Consider the unique bounded martingale U defined on $\tilde{\mathscr{H}}$ by the equation*

$$dU=(dA^\dagger-dA)\,U,\ U(0)=1.$$

Then

(i) $\{e^{-t/2}U(t)\}$ *is a unitary adapted process;*

(ii) $U(t)$ *leaves $\tilde{\mathscr{E}}$ invariant, where $\tilde{\mathscr{E}}$ is defined with respect to the pair $(\mathscr{h}_0, L_2(\mathbb{R}_+))$.*

Proof. In the notations of [1], $e^{-t/2}U(t) = 1_0 \otimes W(\chi_{[0,t]}, 1)$, where 1_0 is the identity in $\mathscr{h}_0$, W is the Weyl representation of the Euclidean group over $L_2(\mathbb{R}_+)$, and

$$W(\chi_{[0,t]},\ 1)\,\psi(f) = \exp\left(-\frac{t}{2} - \int_0^t f(s)\,ds\right)\psi(f + \chi_{[0,t]}).$$

Thus both (i) and (ii) are fulfilled. ∎

PROPOSITION 3.8. *Let X be a bounded regular martingale in $\tilde{\mathscr{H}}$ and let L_X, K_X be the associated adapted processes defined in Proposition 3.4 and Corollary 3.5. Set*

$$Y(t) = X(t)\,U(t) - \int_0^t K_X^\dagger(s)\,U(s)\,ds \tag{3.10}$$

where U is the martingale defined in Proposition 3.7. Then

(i) *Y is a bounded regular martingale;*

(ii) *there exists a unique bounded adapted process L_Y such that for all $t > a$, $\tilde{u} \in \tilde{\mathscr{H}}_a$,*

$$[Y(t) - Y(a)]\,\tilde{u} = \int_a^t L_Y(s)\,\tilde{u}\,dw(s).$$

Proof. From Corollary 3.5 and Proposition 3.7 it is clear that $Y(t)$ is a bounded operator. Using the pair $(\mathscr{h}_0, L_2(\mathbb{R}_+))$ and the corresponding $\tilde{\mathscr{E}}$ in (2.4) we have from the definitions in Proposition 3.6,

$$Y(t) = Z(t)\,U(t) + S(t)\,U(t) - \int_0^t K_X^\dagger(s)\,U(s)\,ds. \tag{3.11}$$

Since U leaves $\tilde{\mathscr{E}}$ invariant Y is well defined by the above relation on $\tilde{\mathscr{E}}$. Let $t > a$, $u, v \in \mathscr{h}_0$, $f, g \in L_2(\mathbb{R}_+)$. Then by Proposition 3.6(iv) and the martingale property of U we have

$$\begin{aligned}
&\langle u \otimes \psi(f\chi_{[0,a]}),\ Z(t)\,U(t)\,v \otimes \psi(g\chi_{[0,a]})\rangle \\
&\quad= \langle Z^\dagger(t)\,u \otimes \psi(f\chi_{[0,a]}),\ U(t)\,v \otimes \psi(g\chi_{[0,a]})\rangle \\
&\quad= \langle Z^\dagger(a)\,u \otimes \psi(f\chi_{[0,a]}),\ U(a)\,v \otimes \psi(g\chi_{[0,a]})\rangle \\
&\quad= \langle u \otimes \psi(f\chi_{[0,a]}),\ Z(a)\,U(a)\,v \otimes \psi(g\chi_{[0,a]})\rangle.
\end{aligned}$$

Thus $\{Z(t)\,U(t)\}$ is a martingale in $[0, \infty)$ with respect to $(\mathscr{h}_0, L_2(\mathbb{R}_+))$. Next we apply Ito's product formula (2.9) to $S(t)\,U(t)$ to get

$$\begin{aligned}\frac{d}{dt}\,&\langle S^\dagger(t)\,u\otimes\psi(f),\,U(t)\,v\otimes\psi(g)\rangle\\&=\bar{f}(t)\langle\{L_X^\dagger(t)+S^\dagger(t)\}\,u\otimes\psi(f),\,U(t)\,v\otimes\psi(g)\rangle\\&\quad+g(t)\langle\{K_X(t)-S^\dagger(t)\}\,u\otimes\psi(f),\,U(t)\,v\otimes\psi(g)\rangle\\&\quad+\langle K_X(t)\,u\otimes\psi(f),\,U(t)\,v\otimes\psi(g)\rangle.\end{aligned}\tag{3.12}$$

Replacing f and g by $f\chi_{[0,a]}$ respectively and writing (3.12) in integral form we conclude that $S(t)\,U(t)-\int_0^t K_X^\dagger(s)\,U(s)\,ds$ is a martingale in t. Thus Y satisfies the martingale condition with respect to $\tilde{\mathscr{E}}$. In other words Y is a bounded martingale.

For $\tilde{u}\in\tilde{\mathscr{H}}_a$, $t>a$, we have from Corollary 3.5 and Proposition 3.7

$$\begin{aligned}\left\|\int_a^t K_X^\dagger(s)\,U(s)\,\tilde{u}\,ds\right\|^2 &\leqslant \|\tilde{u}\|^2\left(\int_a^t e^{s/2}\,\|K_X(s)\|\,ds\right)^2\\&\leqslant \|\tilde{u}\|^2\,(e^t-e^a)\,\mu([a,t]).\end{aligned}\tag{3.13}$$

On the other hand, by Proposition 3.3(i), Proposition 3.7(i), and (3.3) we have

$$\begin{aligned}\|X(t)\,&U(t)\,\tilde{u}-X(a)\,U(a)\,\tilde{u}\|^2\\&\leqslant 2\,\|X(t)\|^2\,\{\|U(t)\,\tilde{u}\|^2-\|U(a)\,\tilde{u}\|^2\}+2\,\|U(a)\,\tilde{u}\|^2\,\mu([a,t])\\&\leqslant 2\,\|\tilde{u}\|^2\,\{e^t\,\|X(t)\|^2-e^a\,\|X(a)\|^2+e^a\mu([a,t])\}.\end{aligned}\tag{3.14}$$

Inequalities (3.13), (3.14) and similar estimates for $Y^\dagger(t)$ show that Y is a regular martingale. Proposition 3.4 and the remark after Corollary 3.5 show the existence of L_Y with property (ii). ∎

PROPOSITION 3.9. *Let X be a bounded regular martingale and let Z, $Z^\dagger$, U, Y, L_X, K_X, L_Y be defined as in Propositions* 3.6–3.8. *Set*

$$M(t)=L_Y(t)\,U(t)^{-1}-X(t)-L_X(t).$$

Then the vector process $\eta(t,u,f)$ defined in Proposition 3.6(iii) *satisfies the relation*

$$f(t)\,\eta(t,u,f)=f(t)\{Z(t)\,u\otimes\psi(f\chi_{[0,t]})+M^\dagger(t)\,u\otimes\psi(f\chi_{[0,t]})\}\qquad a.e.\ t.$$

Furthermore Z satisfies the equation

$$Z(t)=X(0)+\int_0^t M(s)\,dA(s)$$

on $\tilde{\mathscr{E}}$ defined with respect to the pair $(\mathscr{h}_0, \mathscr{M})$, where $\mathscr{M}$ is the linear manifold of all locally bounded functions in $L_2(\mathbb{R}_+)$.

Proof. For $t>a$, $u, v \in \mathscr{h}_0$, $f, g \in L_2(\mathbb{R}_+)$, we have from Proposition 3.8(ii), (3.1), and isometry of Ito integrals

$$\langle u \otimes \psi(f\chi_{[0,t]}), [Y(t) - Y(a)]\, v \otimes \psi(g\chi_{[0,a]}) \rangle$$
$$= \int_a^t \bar{f}(s) \langle u \otimes \psi(f\chi_{[0,s]}), L_Y(s)\, v \otimes \psi(g\chi_{[0,a]}) \rangle\, ds. \tag{3.15}$$

Using (3.11), (3.12), and (3.15) we get

$$\langle u \otimes \psi(f\chi_{[0,t]}), \{Z(t)\, U(t) - Z(a)\, U(a)\}\, v \otimes \psi(g\chi_{[0,a]} \rangle$$
$$= \int_a^t \bar{f}(s) \langle u \otimes \psi(f\chi_{[0,s]}), \{L_Y(s) - (L_X(s) + S(s))\, U(s)\}$$
$$\times v \otimes \psi(g\chi_{[0,a]}) \rangle\, ds$$
$$= \int_a^t \bar{f}(s) \langle u \otimes \psi(f\chi_{[0,s]}), \{L_Y(s) - (L_X(s) + X(s) - Z(s))\, U(s)\}$$
$$\times v \otimes \psi(g\chi_{[0,a]}) \rangle\, ds$$
$$= \int_a^t \bar{f}(s) \langle u \otimes \psi(f\chi_{[0,s]}), (M(s) + Z(s))\, U(s)$$
$$\times v \otimes \psi(g\chi_{[0,a]}) \rangle\, ds. \tag{3.16}$$

On the other hand, we have

$$U(t)\, v \otimes \psi(g\chi_{[0,a]}) = e^{t/2} v \otimes W(\chi_{[0,t]}, I)\, \psi(g\chi_{[0,a]})$$
$$= \exp\left(-\int_0^a g(\tau)\, d\tau\right) v \otimes \psi(g\chi_{[0,a]} + \chi_{[0,t]})$$

and hence by Proposition 3.6(iii) and (3.1),

$$\frac{d}{dt} \langle Z^{\dagger}(t)\, u \otimes \psi(f\chi_{[0,t]}), U(t)\, v \otimes \psi(g\chi_{[0,a]}) \rangle$$
$$= \frac{d}{dt} \left\langle X^{\dagger}(0)\, u \otimes \Omega + \int_0^t f(s)\, \eta(s, u, f)\, dw(s), e^{-\int_0^a g(\tau)\, d\tau} \right.$$
$$\left. \times v \otimes \psi(g\chi_{[0,a]} + \chi_{[0,t]}) \right\rangle$$

$$= \frac{d}{dt} \int_0^t \bar{f}(s)(g(s)\,\chi_{[0,a]}(s)+1)\, e^{-\int_0^a g(\tau)\,d\tau}$$

$$\times \langle \eta(s, u, f), v \otimes \psi(g\chi_{[0,a]} + \chi_{[0,t]}) \rangle \, ds$$

$$= \bar{f}(t)\, e^{-\int_0^a g(\tau)\,d\tau} \langle \eta(t, u, f), v \otimes \psi(g\chi_{[0,a]} + \chi_{[0,t]}) \rangle$$

$$= \bar{f}(t) \langle \eta(t, u, f), U(t)\, v \otimes \psi(g\chi_{[0,a]}) \rangle \qquad \text{a.e. } t > a. \tag{3.17}$$

Comparing (3.16) and (3.17) and using the totality of the set $\{v \otimes \psi(g\chi_{[0,a]}),\ v \in \mathscr{h}_0,\ g \in L_2(\mathbb{R}_+),\ 0 < a < t\}$ in $\mathscr{H}_t$ we conclude that

$$f(t)\, U^\dagger(t)\, \eta(t, u, f) = f(t)\, U^\dagger(t)\{M^\dagger(t) + Z^\dagger(t)\}\, u \otimes \psi(f\chi_{[0,t]}) \qquad \text{a.e. } t,$$

which proves the first part.

Once again by Proposition 3.6(iii), (3.1), isometry of Ito integrals, and the first part of this proposition we have

$$\langle Z^\dagger(t)\, u \otimes \psi(f\chi_{[0,t]}), v \otimes \psi(g\chi_{[0,t]}) \rangle$$

$$= \langle X^\dagger(0)\, u \otimes \Omega, v \otimes \Omega \rangle + \int_0^t \bar{f}(s)\, g(s) \langle \eta(s, u, f), v \otimes \psi(g\chi_{[0,s]}) \rangle \, ds$$

$$= \langle u \otimes \Omega, X(0)\, v \otimes \Omega \rangle$$

$$+ \int_0^t \bar{f}(s)\, g(s) \langle u \otimes \psi(\bar{f}\chi_{[0,s]}), (Z(s) + M(s))\, v \otimes \psi(g\chi_{[0,s]}) \rangle \, ds.$$

Since $\psi(f) = \psi(f\chi_{[0,t]}) \otimes \psi(f\chi_{[t,\infty)})$ the adaptedness of X, Z, M implies

$$\langle u \otimes \psi(f), Z(t)\, v \otimes \psi(g) \rangle$$

$$= \langle u \otimes \psi(f), X(0)\, v \otimes \psi(g) \rangle \exp\left(-\int_0^t \bar{f}(s)\, g(s)\, ds \right)$$

$$+ \left(\exp \int_t^\infty \bar{f}(s)\, g(s)\, ds \right) \int_0^t \bar{f}(s)\, g(s)$$

$$\times \langle u \otimes \psi(f\chi_{[0,s]}), (Z(s) + M(s))\, v \otimes \psi(g\chi_{[0,s]}) \rangle \, ds.$$

Differentiating with respect to t and simplifying we get

$$\frac{d}{dt} \langle u \otimes \psi(f), Z(t)\, v \otimes \psi(g) \rangle$$

$$= \bar{f}(t)\, g(t) \langle u \otimes \psi(f), M(t)\, v \otimes \psi(g) \rangle \qquad \text{a.e. } t. \tag{3.18}$$

From Proposition 3.4 and the definition of M we obtain

$$
\begin{aligned}
&\|M(s)\, u \otimes \psi(f)\|^2 \\
&\quad \leqslant 3\{\|X(s)\|^2 + e^{-s}\, \|L_Y(s)\|^2 + \|L_X(s)\|^2\}\, \|u \otimes \psi(f)\|^2 \\
&\quad \leqslant 3\{\|X(s)\|^2 + e^{-s} v'(s) + \mu'(s)\}\, \|u \otimes \psi(f)\|^2
\end{aligned} \tag{3.19}
$$

where v', μ' are the derivatives of the absolutely continuous Radon measures v, μ respectively with respect to which Y and X are regular martingales. By proposition 3.8(i) and (3.19) we have for any $u \in \mathscr{h}_0$, $f \in \mathscr{M}$,

$$
\int_0^t |f(s)|^2\, \|M(s)\, u \otimes \psi(f)\|^2\, ds < \infty \qquad \text{for all } t.
$$

In other words M is a bounded adapted process admitting the stochastic integral $\int_0^t M(s)\, dA(s)$ with respect to the pair $(\mathscr{h}_0, \mathscr{M})$ and a comparison of (3.18) with (2.7) shows that

$$
dZ = M\, dA,\ Z(0) = X(0) \qquad \text{with respect to } (\mathscr{h}_0, \mathscr{M}). \quad \blacksquare
$$

We are now ready to state the main result on the representation of a bounded regular martingale as a stochastic integral.

THEOREM 3.10. *Let X be a bounded martingale on $\mathscr{H} = \mathscr{h}_0 \otimes \Gamma(L_2(\mathbb{R}_+))$ which is regular with respect to a Radon measure μ on $\mathbb{R}_+$ and let $\mathscr{M} \subset L_2(\mathbb{R}_+)$ be the linear manifold of locally bounded functions. Then there exist three bounded adapted processes K, L, M such that*

$$
dX = M\, dA + K^\dagger\, dA + L\, dA^\dagger \tag{3.20}
$$

with respect to the pair $(\mathscr{h}_0, \mathscr{M})$ and

$$
\max(\|K(s)\|^2, \|L(s)\|^2) \leqslant \mu'_{\mathrm{ac}}(s) \qquad \text{for all } s
$$

where μ'_{ac} denotes the density of the absolutely continuous part of μ. Such a triple (K, L, M) is unique modulo a set of Lebesgue measure zero.

Conversely, if a bounded martingale X admits a representation (3.20) *with respect to $(\mathscr{h}_0, \mathscr{M})$, where K, L, M are bounded adapted processes and $\|K(\cdot)\|$, $\|L(\cdot)\|$ are locally square integrable then X is regular.*

Proof. The first part is immediate from the definition of Z in Proposition 3.6 and Proposition 3.9. Uniqueness follows from the formulae for L_X and K_X in Proposition 3.4, Corollary 3.5, and the definition of M in Proposition 3.9. The converse part is a restatement of Proposition 3.2. $\blacksquare$

4. Applications

Now we look at a few examples of regular bounded martingales and their representations. Throughout this section, we shall denote by $\mathcal{M}$ the linear manifold of locally bounded functions in $L_2(\mathbb{R}_+)$.

THEOREM 4.1. *Let U be a unitary martingale on $\tilde{\mathcal{H}} = \hbar_0 \otimes \Gamma(L_2(\mathbb{R}_+))$. Then there exists a unique unitary adapted process W such that*

$$dU = (W - 1)\, U\, d\Lambda$$

with respect to the pair $(\hbar_0, L_2(\mathbb{R}_+))$.

Proof. Let $t > a$, $\tilde{u} \in \tilde{\mathcal{H}}_a$. Then by (3.3)

$$\|[U(t) - U(a)]\, \tilde{u}\|^2 = \|U(t)\, \tilde{u}\|^2 - \|U(a)\, \tilde{u}\|^2 = 0.$$

Similarly $\|[U^\dagger(t) - U^\dagger(a)]\, \tilde{u}\|^2 = 0$. In other words U is regular with respect to the Radon measure 0. By Theorem 3.10, $K_U = L_U = 0$ and we have the representation

$$dU = M\, d\Lambda \qquad \text{with respect to } (\hbar_0, \mathcal{M}).$$

Applying Ito's product formula (2.9) we obtain

$$U^\dagger M + M^\dagger U + M^\dagger M = MU^\dagger + UM^\dagger + MM^\dagger = 0.$$

In other words $M + U$ is a unitary adapted process. Writing $M = (W - 1)\, U$ we obtain the required result. ∎

The next example is motivated by the Weyl representation introduced in [1].

THEOREM 4.2. *Let $\hbar_0 = \mathbb{C}$, $f \in L_2(\mathbb{R}_+)$, and let $W_f = \{W_f(t)\}$ be a unitary adapted process such that the process U defined by*

$$U(t) = \left(\exp \tfrac{1}{2} \int_0^t |f(s)|^2\, ds\right) W_f(t) \tag{4.1}$$

is a martingale. Then there exists a unitary and an isometric adapted process J_f and L_f respectively such that

$$dW_f = \{(J_f - 1)\, d\Lambda - \bar{f}(t)\, L_f^\dagger J_f\, dA + f(t)\, L_f\, dA^\dagger - \tfrac{1}{2}\, |f(t)|^2\, dt\}\, W_f \tag{4.2}$$

with respect to the pair $(\mathbb{C}, L^2(\mathbb{R}_+))$.

Proof. Let $t > a$, $u \in \tilde{\mathscr{H}}_a$. Then

$$\begin{aligned}\|U(t)\,u - U(a)\,u\|^2 &= \|U(t)\,u\|^2 - \|U(a)\,u\|^2\\ &= \|u\|^2 \left\{\exp \int_0^t |f(s)|^2\,ds - \exp \int_0^a |f(s)|^2\,ds\right\}\\ &= \|U^\dagger(t)\,u\|^2 - \|U^\dagger(a)\,u\|^2,\end{aligned}$$

showing that U is a bounded regular martingale. By Theorem 3.10

$$dU = M'\,d\Lambda + K'^\dagger\,dA + L'\,dA^\dagger$$

with respect to $(\mathbb{C}, \mathscr{M})$, where M', K', L' are bounded adapted processes. Now (4.1) implies

$$dW_f = M\,d\Lambda + K^\dagger\,dA + L\,dA^\dagger - \tfrac{1}{2}\,|f(t)|^2\,W_f\,dt \tag{4.3}$$

where M, K, L are again bounded adapted processes. Using Ito's product formula (2.9) on the equations $W_f^\dagger W_f = W_f W_f^\dagger = 1$ we obtain

$$\begin{aligned}W_f^\dagger M + M^\dagger W_f + M^\dagger M = W_f M^\dagger + M W_f^\dagger + M M^\dagger &= 0,\\ W_f^\dagger K^\dagger + L^\dagger W_f + L^\dagger M = W_f K + L W_f^\dagger + MK &= 0,\\ L^\dagger L = K^\dagger K = |f(t)|^2 \qquad \text{a.e. } t.\end{aligned} \tag{4.4}$$

Define

$$L_f(t) = f(t)^{-1}\,L(t)\,W_f^\dagger(t) \qquad \text{if} \quad f(t) \neq 0; \; = 1 \text{ otherwise.}$$

The last equation in (4.4) implies that $L_f(t)$ is an isometry. The first two equations in (4.4) imply that $W_f + M$ is a unitary adapted process. Putting $W_f + M = J_f W_f$ we obtain from the third and fourth equations in (4.4),

$$K^\dagger(t) = -\bar{f}(t)\,L_f^\dagger(t)\,J_f(t)\,W_f(t).$$

Substituting these expressions in (4.3) we get (4.2). ∎

We now study the example of a Hilbert–Schmidt martingale X, where $X(t) = X_t^0 \otimes 1^t$ in $\tilde{\mathscr{H}}$, X_t^0 being a Hilbert–Schmidt operator in $\tilde{\mathscr{H}}_t$. We write

$$\alpha_X(t) = \|X_t^0\|_2^2 \tag{4.5}$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm in $\tilde{\mathscr{H}}_t$.

PROPOSITION 4.3. *Let X be a Hilbert–Schmidt martingale in $\tilde{\mathscr{H}}$. Then* (i) *the function α_X defined by* (4.5) *is a continuous increasing function*; (ii) *X is* *regular with respect to the Radon measure μ_X which is the absolutely continuous part of the Stieltjes' measure generated by α_X.*

Proof. Let $t > a$, $\tilde{u} \in \tilde{\mathscr{H}}_a$, $\tilde{u} \neq 0$. Choose $\{\tilde{u}_i\}$ *to be a complete orthonormal basis* $\tilde{\mathscr{H}}_a$ *with* $\tilde{u}_1 = \|\tilde{u}\|^{-1} \tilde{u}$. *Then*

$$\alpha_X(t) = \|X_t^0\|_2^2 \geqslant \sum_i \|(X_t^0 \tilde{u}_i) \otimes \Omega^{[a,t]}\|^2,$$

$$\alpha_X(a) = \|X_a^0\|_2^2 = \sum_i \|X_a^0 \tilde{u}_i\|^2.$$

Thus

$$\begin{aligned}\|\tilde{u}\|^{-2} \|[X(t) - X(a)]\, \tilde{u}\|^2 &\leqslant \sum_i \|[X(t) - X(a)]\, \tilde{u}_i\|^2 \\ &= \sum_i \|(X_t^0 \tilde{u}_i) \otimes \Omega^{[a,t]}\|^2 - \sum_i \|X_a^0 \tilde{u}_i\|^2 \\ &\leqslant \alpha_X(t) - \alpha_X(a)\end{aligned}$$

which shows that α_X is increasing. As in [4], $\{X(t)\}$ can be isometrically identified as a classical $\mathscr{h}_0 \otimes \mathscr{h}_0$ valued martingale adapted to a pair of independent standard Brownian motions and hence α_X is continuous in t. This proves (i).

To prove (ii) note that (i) implies

$$\|[X(t) - X(a)]\, \tilde{u}\|^2 \leqslant \|\tilde{u}\|^2 (\alpha_X(t) - \alpha_X(a))$$

and a similar inequality for the Hilbert–Schmidt martingale $X^\dagger$. By Proposition 3.3(ii) the required result follows. ∎

THEOREM 4.4. *Every Hilbert–Schmidt martingale X in $\tilde{\mathscr{H}}$ admits the representation*

$$dX = -X\, dA + K^\dagger\, dA + L\, dA^\dagger$$

with respect to $(\mathscr{h}_0, L_2(\mathbb{R}_+))$, *where* K, L *are Hilbert–Schmidt adapted processes.*

Proof. Using the orthonormal basis $\{\tilde{u}_i\}$ introduced in the proof of Proposition 4.3 and the definition of L in Proposition 3.4 we have

$$\begin{aligned}\sum_i \int_a^t \|L(s)\, \tilde{u}_i\|^2\, ds &= \sum \|[X(t) - X(a)]\, \tilde{u}_i\|^2 \\ &\leqslant \alpha_X(t) - \alpha_X(a) \qquad \text{for} \quad t > a,\end{aligned}$$

so that $L(s)$ is Hilbert–Schmidt in $\tilde{\mathscr{H}}_s$ and

$$\|L(s)\|_2^2 \leqslant \alpha'_X(s) \qquad \text{a.e. } s.$$

Similarly, from Corollary 3.5 we get

$$\|K(s)\|_2^2 \leqslant \alpha'_X(s) \qquad \text{a.e. } s.$$

By Proposition 4.3(ii) and Theorem 3.10 it follows that X admits the representation

$$dX = M\,d\Lambda + K^\dagger\,dA + L\,dA^\dagger$$

with respect to $(\mathscr{h}_0, \mathscr{M})$, where K, L, and hence by Proposition 3.9, M are adapted Hilbert–Schmidt processes. We have from (2.7), for u, $v \in \mathscr{h}_0$, $f, g \in \mathscr{M}$,

$$\begin{aligned}
&\langle u \otimes \psi(f), X(t)\, v \otimes \psi(g)\rangle \\
&\quad = \langle u \otimes \psi(f), X(0)\, v \otimes \psi(g)\rangle \\
&\qquad + \int_0^t \{\bar{f}(s)\, g(s) \langle u \otimes \psi(f), M(s)\, v \otimes \psi(g)\rangle \\
&\qquad + \bar{f}(s) \langle u \otimes \psi(f), L(s)\, v \otimes \psi(g)\rangle \\
&\qquad + g(s) \langle K(s)\, u \otimes \psi(f), v \otimes \psi(g)\rangle\}\, ds. \qquad (4.6)
\end{aligned}$$

Define the rank one operator P_t^0 in $\tilde{\mathscr{H}}_t$ by

$$P_t^0 \tilde{u} = \langle v \otimes \psi(g\chi_{[0,t]}), \tilde{u} > u \otimes \psi(f\chi_{[0,t]}),\ \tilde{u} \in \tilde{\mathscr{H}}_t.$$

Then (4.6) implies after an elementary computation

$$\begin{aligned}
\frac{d}{dt} \operatorname{tr} P_t^{0\dagger} X_t^0 &= \frac{d}{dt} \langle u \otimes \psi(f\chi_{[0,t]}), X(t)\, v \otimes \psi(g\chi_{[0,t]})\rangle \\
&= \operatorname{tr} P_t^{0\dagger} \{\bar{f}(t)\, g(t)(X_t^0 + M_t^0) + \bar{f}(t)\, L_t^0 + g(t)\, K_t^{0\dagger}\}. \qquad (4.7)
\end{aligned}$$

We now use the isometric isomorphism between the Hilbert space of Hilbert–Schmidt operators in a Hilbert space $\mathscr{h}$ and the tensor product Hilbert space $\mathscr{h} \otimes \mathscr{h}$. Then $X(t)$ can be identified with a square integrable $\mathscr{h}_0 \otimes \mathscr{h}_0$ valued classical martingale adapted to the product filtration of a pair of independent standard Brownian motions. Then by Proposition 3.1 adapted to such a pair we conclude that

$$X_t^0 = \int_0^t \xi_X\, dw_1 + \int_0^t \eta_X\, dw_2$$

where ξ_X and η_X are $h_0 \otimes h_0$ valued square integrable nonanticipating Brownian functionals with respect to (w_1, w_2). The rank one martingale $P(t) = P_t^0 \otimes 1^t$ satisfies

$$P_t^0 = P_0^0 + \int_0^t f(s)\, P_s^0\, dw_1 + \int_0^t \bar{g}(s)\, P_s^0\, dw_2$$

where P_0^0 is the rank one operator in h_0 determined by $u \otimes v$ in $h_0 \otimes h_0$ but viewed as a constant random variable. By the isometry of classical Ito integrals we obtain

$$\frac{d}{dt} \operatorname{tr} P_t^{0\dagger} X_t^0 = \bar{f}(t) \operatorname{tr} P_t^{0\dagger} \xi_X(t) + g(t) \operatorname{tr} P_t^{0\dagger} \eta_X(t) \qquad \text{a.e. } t. \tag{4.8}$$

Comparing (4.7) and (4.8) we conclude that $X_t^0 + M_t^0 = 0$. Thus

$$dX = -X\, d\Lambda + K^\dagger\, dA + L\, dA^\dagger \qquad \text{with respect to } (h_0, \mathscr{M}).$$

Since $\|X(t)\|$ is increasing in t it follows that the same representation holds with respect to $(h_0, L_2(\mathbb{R}_+))$. ∎

The last example is that of a representation of canonical anticommutation relations (CAR) or Fermion annihilation and creation operators over $L_2(\mathbb{R}_+)$. From [5] we recall that one such representation was obtained as follows.

Let $h_0 = \mathbb{C}$ so that $\tilde{\mathscr{H}} = \mathscr{H}$. Define the unitary operators $J(s)$ on $\mathscr{H}$ through the relations

$$J(s)\, \psi(f) = \psi(-f\chi_{[0,s]} + f\chi_{(s,\infty)}), \qquad f \in L_2(\mathbb{R}_+).$$

Indeed, $J(s)$ is the second quantization of the reflection operator $f \to -f\chi_{[0,s]} + f\chi_{(s,\infty)}$ on $L_2(\mathbb{R}_+)$. Then $J(s) = J(s)^\dagger = J(s)^{-1}$ and J is a unitary adapted process. Define for any $\phi \in L_2(\mathbb{R}_+)$

$$\begin{aligned} F_\phi(t) &= \int_0^t \bar{\phi}(s)\, J(s)\, dA(s) \\ F_\phi^\dagger(t) &= \int_0^t \phi(s)\, J(s)\, dA^\dagger(s). \end{aligned} \tag{4.9}$$

$\{F_\phi, F_\phi^\dagger\}$ are adjoint pairs of bounded martingales satisfying the CAR: for $\phi, \psi \in L_2(\mathbb{R}_+)$,

$$\begin{aligned} [F_\phi(t), F_\psi(t)]_+ &\equiv F_\phi(t)\, F_\psi(t) + F_\psi(t)\, F_\phi(t) = 0, \\ [F_\phi(t), F_\psi^\dagger(t)]_+ &= \int_0^t \bar{\phi}(s)\, \psi(s)\, ds. \end{aligned} \tag{4.10}$$

Also the algebra generated by $\{F_\phi(t), F_\psi^\dagger(t), \phi, \psi \in L_2(\mathbb{R}_+)\}$ is irreducible in $\mathscr{H}_t$ and $F_\phi(t)\,\Omega = 0$.

We now address ourselves to the question whether there are any representations other than (4.9) of the CAR (4.10). That this is unique up to a multiplicative phase factor is the content of the next theorem.

THEOREM 4.5. *Let $h_0 = \mathbb{C}$ and let $\{X_\phi(t), X_\phi^\dagger(t)\}$ be an adjoint pair of martingales with respect to the pair $(\mathbb{C}, L_2(\mathbb{R}_+))$ satisfying the* CAR (4.10) *for $\phi \in L_2(\mathbb{R}_+)$. Furthermore let the algebra $\mathcal{A}_t$ generated by $\{X_\phi(t), X_\psi^\dagger(t) \mid \phi, \psi \in L_2(\mathbb{R}_+)\}$ be irreducible in $\mathscr{H}_t$ and $X_\phi(t)\,\Omega = 0$ for each $t > 0$. Then there exists a Borel function θ on $\mathbb{R}_+$ of modulus unity such that*

$$X_\phi(t) = F_{\theta\phi}(t) \qquad \text{for all} \quad \phi \in L_2(\mathbb{R}_+),\ t \geqslant 0.$$

Proof. It is clear from CAR that X_ϕ and $X_\phi^\dagger$ are bounded adapted processes. Using the martingale property and CAR we have for any $t > a$, $u \in \mathscr{H}_a$,

$$\|[X_\phi(t) - X_\phi(a)]\,u\|^2 + \|[X_\phi^\dagger(t) - X_\phi^\dagger(a)]\,u\|^2 = \|u\|^2 \int_a^t |\phi(s)|^2\,ds.$$

In other words X_ϕ is regular with respect to the Radon measure μ defined by $\mu([s,t]) = \int_s^t |\phi(\tau)|^2\,d\tau$ for all $0 < s < t < \infty$. By Theorem 3.10, $dX_\phi = M_\phi\,dA + K_\phi^\dagger\,dA + L_\phi\,dA^\dagger$ with respect to $(\mathbb{C}, \mathscr{M})$, where M_ϕ, K_ϕ, L_ϕ are bounded adapted processes with

$$\max(\|K_\phi(t)\|^2, \|L_\phi(t)\|^2) \leqslant |\phi(t)|^2.$$

Since CAR implies that for scalars a, b and $\phi, \psi \in L_2(\mathbb{R}_+)$,

$$[X_{a\phi + b\psi}(t) - \bar{a}X_\phi(t) - \bar{b}X_\psi(t),\ X_{a\phi+b\psi}^\dagger(t) - aX_\phi^\dagger(t) - bX_\psi^\dagger(t)]_+ = 0$$

it follows that the map $\phi \to X_\phi(t)$ is antilinear in ϕ. Hence

$$\begin{aligned} dX_\phi &= \bar{\phi}\{M\,dA + K^\dagger\,dA + L\,dA^\dagger\} \\ dX_\phi^\dagger &= \phi\{M\,dA + L^\dagger\,dA + K\,dA^\dagger\} \end{aligned} \tag{4.11}$$

with respect to $(\mathbb{C}, \mathscr{M})$, where M, K, L are bounded adapted processes independent of ϕ, ψ. Applying Ito's product formula (2.9) to $[X_\phi, X_\psi]_+ = 0$ and to $[X_\phi^\dagger, X_\psi]_+ = \int_0^t \phi\bar{\psi}\,ds$, we get

$$\begin{aligned} \bar{\phi}[M, X_\psi]_+ + \bar{\psi}[M, X_\phi]_+ + 2\bar{\phi}\bar{\psi}M^2 &= 0 \\ \bar{\phi}[K^\dagger, X_\psi]_+ + \bar{\psi}[K^\dagger, X_\phi]_+ + 2\bar{\phi}\bar{\psi}K^\dagger M &= 0 \\ \bar{\phi}[L, X_\psi]_+ + \bar{\psi}[L, X_\phi]_+ + 2\bar{\phi}\bar{\psi}ML &= 0 \\ 2\bar{\phi}\bar{\psi}K^\dagger L &= 0; \end{aligned} \tag{4.12}$$

$$\begin{aligned}\phi[M^\dagger, X_\psi]_+ + \bar{\psi}[M, X_\phi^\dagger]_+ + \phi\bar{\psi}(M^\dagger M + MM^\dagger) &= 0\\ \phi[L^\dagger, X_\psi]_+ + \bar{\psi}[K^\dagger, X_\phi^\dagger]_+ + \phi\bar{\psi}(L^\dagger M + K^\dagger M^\dagger) &= 0\\ \phi[K, X_\psi]_+ + \bar{\psi}[L, X_\phi^\dagger]_+ + \phi\bar{\psi}(M^\dagger L + MK) &= 0\\ \phi\bar{\psi}(K^\dagger K + L^\dagger L) &= \phi\bar{\psi}.\end{aligned} \tag{4.13}$$

In the first equation of (4.13) choose ψ with supp $\psi \subset [0, a]$. Then for all $t > a$ and for all such ψ, $[M^\dagger, X_\psi]_+ = 0$. Using the antilinearity and continuity of X_ψ in ψ in L_2 norm we conclude that $[M^\dagger, X_\psi]_+ = 0$. Similarly $[M, X_\phi^\dagger]_+ = 0$. Hence $M^\dagger M + MM^\dagger = 0$ or equivalently $M = 0$.

Applying an identical reasoning to the second and third equations of (4.12) we get

$$[L, X_\phi]_+ = [L^\dagger, X_\phi]_+ = [K, X_\phi]_+ = [K^\dagger, X_\phi]_+ = 0 \tag{4.14}$$

while the last equations of (4.12) and (4.13) give

$$K^\dagger L = 0, \qquad K^\dagger K + L^\dagger L = 1. \tag{4.15}$$

Let $R(t) = K(t) + L(t)$ so that $R^\dagger R = 1$. From (4.14) it follows that $[R, X_\phi]_+ = [R, X_\phi^\dagger]_+ = 0$. Hence $RR^\dagger$ commutes with X_ϕ and $X_\phi^\dagger$ for all $\phi \in L_2(\mathbb{R}_+)$. The irreducibility of $\mathcal{A}_t$ in $\mathcal{H}_t$ and the property that $RR^\dagger$ is a nonzero projection imply that $RR^\dagger = 1$. Similarly R^2 commutes with X_ϕ and $X_\phi^\dagger$ and hence $R^2(t) = \rho^2(t)$, where $\rho(t)$ is a scalar of modulus unity. By the same arguments there exist scalars $\alpha(t)$, $\beta(t)$ such that $KR^\dagger = \alpha$, $LR^\dagger = \beta$. Using (4.15) we conclude

$$\begin{aligned} K(t) &= \alpha(t)\, R(t), & L(t) &= \beta(t)\, R(t),\\ \bar{\alpha}(t)\, \beta(t) &= 0, & |\alpha|^2 + |\beta|^2 &= 1. \end{aligned} \tag{4.16}$$

By (4.11)

$$dX_\phi = \bar{\phi}\{\bar{\alpha} R^\dagger\, dA + \beta R\, dA^\dagger\}. \tag{4.17}$$

Using Ito's product formula (2.9)

$$0 = \|X_\phi(t)\, \Omega\|^2 = \int_0^t |\phi|^2\, |\beta|^2\, ds.$$

Thus $\beta = 0$. Combining (4.16) and (4.17) we get

$$dX_\phi = \bar{\phi}\bar{\varepsilon} S\, dA \tag{4.18}$$

where $|\varepsilon(t)|=1$ and S is a reflection valued adapted process. Since $\{F_\phi(t), F_\psi^\dagger(t)\}$ and $\{X_\phi(t), X_\psi^\dagger(t)\}$ satisfy CAR (4.10), act irreducibly, and annihilate the vacuum Ω_t in $\mathscr{H}_t$ it follows that there exists a unique unitary operator $U(t)$ in $\mathscr{H}_t$ such that

$$X_\phi(t)\, U(t) = U(t)\, F_\phi(t), \qquad U(t)\, \Omega_t = \Omega_t \tag{4.19}$$

where $U(t)$ is interpreted as the operator $U(t) \otimes 1'$ in $\mathscr{H}$. Denote by U the unitary adapted process defined by $\{U(t)\}$. For any $t > s$ and $\phi_i \in L_2(\mathbb{R}_+)$, $1 \leqslant i \leqslant n$,

$$\begin{aligned} F_{\phi_1}^\dagger(s) \cdots F_{\phi_n}^\dagger(s)\, \Omega &= U^\dagger(s)\, X_{\phi_1}^\dagger(s) \cdots X_{\phi_n}^\dagger(s)\, \Omega \\ &= F_{\phi_1 \chi_{[0,s]}}^\dagger(t) \cdots F_{\phi_n \chi_{[0,s]}}^\dagger(t)\, \Omega = U^\dagger(t)\, X_{\phi_1}^\dagger(s) \cdots X_{\phi_n}^\dagger(s)\, \Omega. \end{aligned}$$

This shows that U is a unitary martingale and hence by Theorem 4.1.,

$$dU = (W-1)\, U\, dA \tag{4.20}$$

with respect to $(\mathbb{C}, L_2(\mathbb{R}_+))$, where W is another unitary adapted process. By Ito's product formula (2.9) applied to (4.19) and using (4.18), (4.20), (4.9), we get

$$\begin{aligned} X_\phi (W-1)\, U &= (W-1)\, U F_\phi = (W-1)\, X_\phi U \\ \bar{\phi}\bar{\varepsilon} S W U &= \bar{\phi} U J. \end{aligned} \tag{4.21}$$

The first equation in (4.21) implies that W commutes with X_ϕ and hence with $X_\phi^\dagger$. By irreducibility $W(t) = \theta(t)$, where $\theta(t)$ is a scalar of modulus unity. Substituting this in (4.20) and solving it with initial condition $U(0) = 1$ we obtain

$$U(t) = \Gamma(\theta \chi_{[0,t]} + \chi_{(t,\infty)})$$

where the right-hand side denotes the second quantization of multiplication by $\theta \chi_{[0,t]} + \chi_{(t,\infty)}$. In particular U commutes with J and the second equation in (4.21) implies that $S(t) = \varepsilon(t)\, \overline{\theta(t)}\, J(t)$. Substituting this in (4.18) we get $dX_\phi = \bar{\phi}\bar{\theta} J\, dA$. In other words $X_\phi(t) = F_{\theta\phi}(t)$ for all $\phi \in L_2(\mathbb{R}_+)$.

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